

Second Order Necessary conditions in Non-linear Programming

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Abstract. In this paper, we consider conditions which are necessary for a directionally differentiable function to have an unconstrained local minimum at a point $x_0 \in X$ where X is a normed linear space. The necessary conditions are expressed in terms of first order directional derivatives of non-linear analysis and certain second order directional derivatives.

1. Introduction We consider a function $f: X \rightarrow \mathbb{R}$ which is continuous and directionally differentiable on a non-empty open set S . We give a necessary condition for f to have a local minimum at a point x_0 of S .

Many authors have provided second order optimality conditions for non-linear optimization problems.

Ben-Tal and Zowe [2] have developed a general theory of second order necessary conditions. However, their frame work and point of view are very different from ours. Our second order conditions are the refinement of the ones given by Chaney[3,4] for semi smooth functions in finite dimensions The concept of second derivatives developed by him are placed entirely within the context of non-smooth analysis as set forth by Clarke[5]. Other notions of second derivatives have been presented by Hirriart-Urruty [6], Auslander [1] and Rockafellar [8]. Of these, Chaney's approach is closest to that of Rockafellar's. We prove the necessary conditions for a directionally differentiable functions in a normed linear space.

In Section 2, we give necessary preliminaries. Section 3 deals with second order necessary conditions for an unconstrained problem.

2.Preliminaries. Throughout the paper X shall stand for a normed linear space and X^* its topological dual. $f: X \rightarrow \mathbb{R}$.

Definition2.1 $f: X \rightarrow \mathbb{R}$ is said to be directionally differentiable at $x_0 \in X$ if the one sided directional derivative of f at x_0 in the direction d defined by
$$\lim_{\lambda \rightarrow 0^+} \frac{f(x_0 + \lambda d) - f(x_0)}{\lambda}$$
 exists for all d in X and the function $x \rightarrow f'(x_0; x)$ is convex and continuous.

Remark2.1 any continuous convex function is locally convex. However, there are non-convex functions which are directionally differentiable.

The following definitions can be seen in [4].

Defdnition2.2 Let $d \in X$. suppose that the sequence $(x_n) \rightarrow x_0$ with $x_n \neq x_0$ for every n . then we say that $x_n \rightarrow x_0$ in in direction d if the

$$\text{sequence } \left(\frac{\|d\|(x_n \rightarrow x_0)}{\|x_n \rightarrow x_0\|} \right) \rightarrow d$$

Definition2.3 Let $d \in X$. the set $\partial_d f(x_0)$ is defined as =

$$\partial_d f(x_0) = \left\{ x^* \in X^* \mid d \text{ and an } x_n^* \in \partial_d f(x_n) \right. \\ \left. \forall n \text{ such that } x_n^* \rightarrow x^* \right\}$$

i.e., it consists of those subgradients of f at x_0 which come from direction d .

Remark2.2 Mifflin has proved that if a function g is semi-smooth at x_0 , then for each direction d , the classical directional derivative $g'(x_0; d)$ exists and is equal to the limit of every sequence $\{ \langle d, x_n^* \rangle \}$

where, for all

$(x_n) \rightarrow x_0$ in direction d and $x_n^* \in \partial g(x_n), \forall n$ converging to x_0^* in $\partial g(x_0)$

Definition 2.4 A multifunction $F: X \rightarrow 2^Y$ is upper semi-continuous if and only if the set

$\{x \in X / F(x) \cap B \neq \emptyset\}$ is closed for every closed subset B of Y .

Definition 2.5 Let $d \in X$. For any $x_0 \in X, f''_-(x_0; d)$ is defined to be

$$f''_-(x_0; d) = \liminf_{\substack{t \rightarrow 0^+ \\ x_n \rightarrow x_0 \text{ in direction } d}} \left\{ \frac{f(x) - f(x_0) - tf'(x_0; d)}{t^2} \right\}$$

and $f''_+(x_0; d)$ is defined by

$$f''_+(x_0; d) = \limsup_{\substack{t \rightarrow 0^+ \\ x_n \rightarrow x_0 \text{ in direction } d}} \left\{ \frac{f(x) - f(x_0) - tf'(x_0; d)}{t^2} \right\}$$

They are respectively called second order lower and upper directional derivatives in the direction d . If both are equal, then the common value is denoted by $f''(x_0; d)$ and is called the second order directional derivative in the direction d .

3. SECOND ORDER NECESSARY CONDITION IN AN UNCONSTRAINED PROBLEM

Before stating the main theorem, we prove the following lemma.

Lemma 3.1 Let $f: X \rightarrow \mathbb{R}$ be a continuous directionally differentiable function on X . Suppose that for each $y \in X$, the function $x \rightarrow f'(x; y)$ is upper semi-continuous. then the multivalued operator $\partial f: X \rightarrow 2^{X^*}$ given by $x \rightarrow \partial f(x)$ is w^* -upper semi-continuous.

Proof We are required to prove that the set

$A = \{x \in X / \partial f(x) \cap B^* \neq \emptyset\}$ is closed for each w^* -closed subset B^* of X^* .

Let $x_0 \in \text{cl.} A$, where $\text{cl.} A$ denotes the closure of A in X . then there exists a sequence $x_n \in A$ such that $(x_n) \rightarrow x_0$. Further, $\partial f(x_n) \cap B^* \neq \emptyset$ for each n . B^* being closed, it is also norm bounded. It follows that B^* is w^* -compact. Hence, (x_n^*) has a convergent subsequence say (x_{n_k}) converging to a point x_0^* in B^* .

By passing onto subsequences if necessary we may suppose without loss of generality that (x_n^*) itself

converges to x_0^* . Since $x_n^* \in \partial f(x_n)$

$$\langle x, x_n^* \rangle \leq f'(x_n; x), \forall x \in X \quad \dots(1)$$

Taking limit as $n \rightarrow \infty$, (1) becomes

$$\lim_{n \rightarrow \infty} \langle x, x_n^* \rangle \leq \limsup_{n \rightarrow \infty} f'(x_n; x) \leq f'(x_0; x) \quad \dots(2)$$

for all $x \in X$, since by assumption, the function $x \rightarrow f'(x; y)$ is upper semi-continuous for each $y \in X$.

As $(x_n^*) \rightarrow x_0^*$, (2) implies

$\langle x, x_0^* \rangle \leq f'(x_0; x)$ which in turn implies that $x_0^* \in \partial f(x_0)$. It follows that $x_0^* \in \partial f(x_0) \cap B^*$. Thus $x_0^* \in A$. That is, A is closed.

Hence the lemma.

Theorem 3.1 Let $f: X \rightarrow \mathbb{R}$ be a function continuous and directionally differentiable on a non-empty open subset S of X . Suppose that $x_0 \in S$ is an unconstrained local minimizer for f over x in S . Let f also satisfy the following two conditions:

$$(i) \quad f'(x_0; d) \geq 0 \Rightarrow f(x_0 + d) \geq f(x_0) \quad \dots(3)$$

(ii) for each y in X , the function $x \rightarrow f'(x; y)$ is upper semi-continuous.(4)

Let $d_0 \in X$ be such that $\|d_0\|=1$ and $f'(x_0; d_0) = 0$. Then given any sequence $(x_n) \rightarrow x_0$ in direction d_0 , there exists a sequence $(x_n^*) \in \partial f(x_n)$ such that the sequence $(x_n^*) \rightarrow 0$. That is, $0 \in \partial f(x_0)$. furthermore, the sequence

$$(f'(x_n; d_0) \rightarrow f'(x_0; d_0) \text{ and } f''(x_0; d_0) \geq 0.$$

Proof Let $\delta_0 > 0$ such that $f(x) \geq f(x_0)$ whenever $x \in B(x_0; \delta_0) \subseteq S$ where

$B(x_0; \delta_0) = \{x \in X \mid \|x - x_0\| < \delta_0\}$. Let $x_n \in S$ be such that (x_n) converges to x_0 in direction d_0 . So, $x_n \in B(x_0; \delta_0)$ for all large n . That is, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $x_n \in B(x_0; \delta_0)$.

Hence $f(x_n) \geq f(x_0)$, for all $n \geq n_0$ (5)

From (4) we have

$$\limsup_{n \rightarrow \infty} f'(x_n; d_0) \leq f'(x_0; d_0) = 0 \dots(6)$$

Claim1 $f'(x_n; d_0) \geq 0$, for all n .

Suppose not. Then there exists an m such that

$$f'(x_m; d_0) < 0$$

From condition(3) it follows that there exists $0 < \lambda_0 < 1$ such that

$$f(x_m + \lambda d_0) - f(x_m) < 0 \dots(7)$$

Now, let a function $b: [0, \lambda_0] \rightarrow \mathbb{R}$ be defined by $b(t) = f(x_m + t d_0)$

b is continuous in t on a compact set $[0, \lambda_0]$. Hence b attains its maximum at a point $t_0 \in [0, \lambda_0]$.

From (7), we have

$$f(x_m + \lambda_0 d_0) = b(\lambda_0) < b(0) = f(x_m) < f(x_m + t_0 d_0) \dots(8)$$

$$f'(x_m + t_0 d_0; d_0) \leq 0 \text{ and } f'(x_m + t_0 d_0; -d_0) \leq 0 \dots(9)$$

As $f'(x_m; \cdot)$ is sublinear, (9) implies that

$$f'(x_m + t_0 d_0; d_0) = 0 \dots(10)$$

$f'(x_m; \cdot)$ being positively homogeneous, and $t_0 < \lambda_0$, (10) gives

$$f'(x_m + t_0 d_0; (\lambda_0 - t_0)d_0) = 0 \dots(11)$$

(3) and (11) imply

$$f(x_m + \lambda_0 d_0) \geq f(x_m + t_0 d_0) \text{ which violates (8)}$$

This contradiction proves the claim, namely,

$$f'(x_n; d_0) \geq 0 \text{ for all } n, \dots(12)$$

Now we choose

$$x_n^* \in \partial f(x_n) \text{ as } \langle d_0, x_n^* \rangle = f'(x_n; d_0) \dots(13)$$

From (4) and Lemma 3.1, we get a subsequence

$$x_{n_k}^* \in \partial f(x_{n_k}), \forall k \text{ such that } x_{n_k}^* \xrightarrow{w^*} x_0^*$$

for some $x_0^* \in \partial f(x_0)$. Hence by passing on to subsequences if necessary, we may suppose that (x_n^*) itself converges to x_0^* . From (6), (12) and (13) we have

$$\langle d_0, x_n^* \rangle \geq 0. \dots(14)$$

$$\text{Hence } \lim_{n \rightarrow \infty} \langle d_0, x_n^* \rangle = \langle d_0, x_0^* \rangle = 0$$

$$\dots(15)$$

$$\text{As } x_o^* \in \partial f(x_o), \quad \langle d_o, x_o^* \rangle \leq f'(x_o; d_o) = 0 \quad \dots(16)$$

(15) and (16) imply that $\langle d_o, x_o^* \rangle = 0$

Claim 2

$$x_o^* = 0$$

Let $g \in X^*$ be such that $g(d_o) = \|d_o\| = 1$ and $0 < \alpha < 1$.

Define a closed convex cone K in X^* by

$$K = \{x \in X / g(x) \geq \alpha \|x\|\}$$

Clearly, $d_o \in \text{int } K$. Give X the order given by K . K is a normal cone with interior points .. $g \in \text{int } K^*$ and hence $[-g, g]$ is a neighbourhood of zero in X^* . $\partial f(x_o)$ being norm bounded $\partial f(x_o) \subseteq [-\lambda g, \lambda g]$, for some $\lambda > 0$.

It follows that $\partial f(x_o)$ has a supremum with respect to K^* , say y_o^* . Then, $\langle x, y_o^* \rangle \geq \langle x, x^* \rangle$ for all $x \in \partial f(x_o)$

In particular,

$$\langle d_o, y_o^* \rangle \geq \langle d_o, x_o^* \rangle,$$

$$\text{That is, } 0 = f'(x_o; d_o) = \langle d_o, x_o^* \rangle = \langle d_o, y_o^* \rangle$$

It follows that $x_o^* = y_o^*$ as $d_o \in \text{int } K$ and

$$x_o^* - y_o^* \text{ is linear.}$$

Thus x_o^* is the supremum of $\partial f(x_o)$ with respect to the order given by K^* .

Since x_o minimizes f over $B(x_o, \delta_o)$, $0 \in \partial f(x_o)$

Hence $x_o^* \geq 0$ and also $\langle d_o, x_o^* \rangle = 0$, $d_o \in \text{int } K$, which implies $x_o^* = 0$ and the claim is proved.

Thus (x_n^*) converges to 0.

As (x_n) converges to x_o in direction d_o by assumption, $x_n^* \in \partial f(x_n)$ for every n , such that (x_n^*) converges to zero.

We have $0 \in \partial_d f(x_o)$ and

$$f''_{-}(x_o; 0; u_o) \geq 0 \text{ and}$$

$$f'(x_o; d) =$$

$$\liminf_{n \rightarrow \infty} \left\{ \frac{f(x_n) - f(x_o) - \langle x_n - x_o, v_n^* \rangle}{t_n^2} \right\}$$

taken over all sequences (x_n) which converge to x_o ,

$$t_n > 0 \text{ for all } n \text{ and } (t_n) \rightarrow 0^+ \text{ such that } \left\{ \frac{x_n - x_o}{t_n} \right\}$$

converges to d_o .

As $f'(x_o; d_o) = 0$ and $f(x_n) - f(x_o) \geq 0$ for sufficiently large n , $f''(x_o; d_o) \geq 0$

This completes the proof of the theorem.

Remark 3.2 Theorem 3.1 generalizes the following theorem of Chaney for semi-smooth functions in [14].

Theorem 3.2 Let $X = \mathbb{R}^n$. Suppose f is semi smooth and locally Lipschitz in an open subset W of \mathbb{R}^n and that x_o is an unconstrained local minimizer for f over W . Suppose that u_o is such that $\|u_o\| = 1$ and $f'(x_o; u_o) = 0$. Then $0 \in \partial f(x_o)$; furthermore

$$f''_{-}(x_o; 0; u_o) \geq 0, \text{ where}$$

$$f''_{-}(x_o; 0; u_o) \geq 0 =$$

$$\liminf_{n \rightarrow \infty} \left\{ \frac{f(x_n) - f(x_o) - \langle x_n - x_o, v_n^* \rangle}{t_n^2} \right\} \text{ taken over}$$

all triples of x_n , t_n and v_n such that

(a) $t_n > 0$ for each n and $(x_n) \rightarrow x_o$, $t_n \rightarrow 0$,

$$(b) \left\{ \frac{x_n - x_0}{t_n} \right\} \rightarrow u_0$$

$$(c) v_n^* \rightarrow v^*, v_n^* \in \partial f(x_n) \text{ for each } n.$$

4.CONCLUSION

Theory of optimization is crowded with various types of first and second order derivatives. The method of proof given here can be applied to more general class of functions. Sufficient conditions can be proved and the results may provide a foundation for the sensitivity analysis of primal and dual problems via Lagrangian.

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